Algebra students may often demonstrate a certain degree of proficiency when manipulating algebraic expressions and verbalizing their behaviors. Do these abilities imply conceptual understanding? What is a reliable indicator that would provide educators with a relatively trustworthy and consistent measure to identify whether students learn algebraic concepts beyond procedures? How might teachers know when the transition from 'operational' or 'process conception' to 'structural' or 'object conception' takes place? Assessing mathematics students' conceptual understanding is critical for educators to make informed decisions when selecting curriculum, planning instruction and developing an assessment program. These decisions should be supported by strong educational foundations and be rooted in theory that provides sustained basis for understanding how mathematics students construct knowledge and how mathematics can be taught. The multiyear research study described in this article attempts to answer these questions. The article introduces a framework for assessing middle school algebra students' levels of understanding of linear equations with one unknown. The framework is rooted in Orit Hazzan and Rina Zazkis's ideas of reducing level of abstraction, Uri Wilensky's and Victor Cifarelli's ideas of the degree of abstraction, Anna Sfard's theory of operational and structural conception, Ed Dubinsky and Michael McDonald's APOS theory, and the concept of representation in mathematics and mathematics education.

THE KEY WORDS: conceptual understanding in algebra, reducing level of abstraction, multiple representations of mathematical relationship.

Introduction

The concept of unknown and operations with unknowns are central to teaching and learning middle school algebra. Learning about linear relationship with one unknown, students may often demonstrate a certain degree of proficiency manipulating algebraic symbols. When encouraged, they can often verbalize and explain the steps they performed, thereby demonstrating awareness of well-known procedures with symbols according to fixed rules. These students show a certain level of "operational conception" (Sfard, 1991, p. 4) or "process conception" (Dubinsky, 1991; Dubinsky & McDonald, 1991, p. 3). It is well known and documented (Herscovics, 1996; Herscovics & Linchevski, 1994; Hiebert, 1988; Hiebert & Carpenter, 1992; Kieran, 1989, 1990, 1992; Kieran & Chalouh, 1993; Langrall & Swafford, 1997) that correct and seemingly fluent demonstration of a procedure does not predictably indicate conceptual understanding or as Skemp (1976) suggested "relational understanding" (p. 21).
In order to make informed decisions and avoid relying on subjective perceptions and feelings whether students learn algebraic concepts beyond procedures, educators are searching for consistent and dependable tools to objectively assess student thinking and behaviours. What would be a reliable indicator that could serve as relatively trustworthy and consistent measure of students’ conceptual understanding? When does the transition from ‘operational’ or ‘process conception’ to ‘structural’ or ‘object conception’ take place? To answer these questions and to extend our knowledge of the processes related to developing conceptual understanding in algebra, the researcher launched a longitudinal study which led to the formation of the three phase ranking framework of conceptual understanding of linear relationship with one unknown described in this paper.

**Background**

Three key ideas served as the foundation of this research study: a) the role of multiple representations in probing understanding of mathematics learning, b) the theory of reducing level of abstraction as a mental process of coping with abstraction level of a given concept or task, and c) the idea of adaptation to abstraction and the development of conceptual understanding as one’s ability to cope with higher levels of abstraction. These ideas guided the researcher’s observations, analysis, and generation of the language that helped communicating some features of the development of conceptual understanding of linear relationship with one unknown. This section briefly outlines major theoretical positions and research that pertain to and serve as a background for this research study.

i) **Algebraic reasoning and conceptual understanding in algebra**

The term algebraic reasoning has been used to describe mathematical processes of generalizing a pattern and modeling problems with various representations (Driscoll, 1999; Herbert & Brown, 1997; NCTM, 2000). Driscoll (1999) defined algebraic reasoning as the “capacity to represent quantitative situations so that relations among variables become apparent” (p. 1). For Langrall and Swafford (1997) algebraic reasoning is “the ability to operate on an unknown quantity as if the quantity is known” (p. 2). Vance (1998) characterized algebraic reasoning as a way of reasoning involving variables, generalizations, different modes of representation, and abstracting from computations. Kaput (1993) viewed algebraic reasoning as a process of construction and representation of patterns and regularities, deliberate generalization, and active exploration and conjecture. These definitions will serve as a basis to explain conceptual understanding in algebra in this paper.

Understanding is a logical power manifested by abstract thought. Piaget (1995) suggested that understanding in general and in mathematics in particular is a highly complex process of abstraction. He proposed the term reflective abstraction (Piaget, 1970, p. 221) to explain the process of developing conceptual understanding. It can be said that those who have a concep-
tual understanding grasp the full meaning of knowledge, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations. Conceptual understanding in algebra can be characterized as the ability to recognize functional relationships between known, and unknown, independent and dependent variables, and to distinguish between and interpret different representations of the algebraic concepts. It is manifested by competency in reading, writing, and manipulating both number symbols and algebraic symbols used in formulas, expressions, equations, and inequalities. Fluency in the language of algebra demonstrated by confident use of its vocabulary and meanings and flexible operation upon its grammar rules (i.e., mathematical properties and conventions) are indicative of conceptual understanding in algebra, as well.

ii) Representations

Representations and symbol systems are fundamental to mathematics as a discipline since mathematics is “inherently representational in its intentions and methods” (Kaput, 1989, p. 169).

Vergnaud (1997) suggested viewing representation as an attribute of mathematical concepts, which are defined by three variables: the situation that makes the concept useful and meaningful, the operation that can be used to deal with the situation, and the set of symbolic, linguistic, and graphic representation that can be used to represent situations and procedures.

Several ideas related to the concept of representation are pertinent to this research. The first and the foremost is that external systems of representation and internal systems of representation and their interaction are essential to mathematics teaching and learning (Goldin & Shteingold, 2001, p. 2).

Internal representations are usually associated with mental images individuals create in their minds. Bruner (1966) proposed to distinguish three different modes of mental representation – the sensory-motor (physical action upon objects), the iconic (creating mental images) and the symbolic (mathematical language and symbols). Estes (1996) posited that internal representation and categorization are the attributes of high-order human cognitive processes; both involve abstraction to represent the entity of the object of communication. Matsuka and Sakamoto (2007) suggested that “By compressing the vast amount of available information, a cognitive process called categorization allows us to process, understand, and communicate complex thoughts and ideas by efficiently utilizing salient and relevant information while ignoring other types” (p.1139). Pape and Tchoshanov (2001) described mathematics representation as an internal abstraction of mathematical ideas or cognitive schemata, that according to Hiebert and Carpenter (1992) the learner constructs to establish internal mental network or representational system. Thus, one can assert that internal representation, categorization and abstraction are closely related mental constructs.

External representations are usually associated with the “knowledge and structure in the environment, as physical
symbols, objects, or dimensions” as well as “external rules, constraints, or relations embedded in physical configurations” (Zhang, 1997, p.180). Goldin and Shteingold (2001) suggested that an external representation “is typically a sign or a configuration or signs, characters, or objects” and that external representation can symbolize “something other than itself” (p. 3). Most of the external representations in mathematics (e.g., signs of operations, symbols or composition of signs and symbols used to represent certain relationships) are conventional; they are objectively determined, defined and accepted (p. 4).

Distinguishing internal representations and external representations, Kaput (1999) used the term “fusion” to emphasize the actions surrounded by the experience of internalizing the external representation. Through classroom discourse and various experiences, teachers facilitate interaction between external representations and the students’ internal representation systems and assist the students in the process of building into their internal mental structure the images of the external representations (Goldin & Shteingold, 2001, p. 2). For instance, to introduce the notion of multiplication, the teacher gives certain meanings and interpretations to a symbol of multiplication (×) as an external representation (or external abstraction, discussed later) that replaces repeated addition symbols (e.g., 4 + 4 + 4 = 4×3). As a result of interaction of “students’ personal symbolization constructs” with the external representation (Goldin & Shteingold, 2001, p. 2), i.e., sign of multiplication, the students build into their mental structure the image of the operation of multiplication which becomes their internal representation. Goldin and Shteingold (2001) stress that the students’ internal representations are affected by their visual imagery, natural language, problem solving abilities and their attitude toward mathematics.

The notion of multiple representations in mathematics education has evolved considerably in recent years, and different theories of representations utilize different terminology (Boulton-Lewis & Tait, 1993; Diezmann, 1999; Diezmann & English, 2001; Goldin & Shteingold, 2001; Moritz, 2000; Outhred & Saradelli, 1997; Swafford & Langrall, 2000; Verschaffel, 1994). Mathematical relationships, principles, and ideas can be expressed in multiple representations including visual representations (i.e. diagrams, pictures, or graphs), verbal representations (written and spoken language) and symbolic representations (numbers, letters). Each type of representation articulates different meanings of mathematical concepts.

Pirie (1998) associated representations with mathematical language which she classified as ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions and quasi-mathematical language. She asserted that the function of any type of representation is to communicate mathematical ideas, and that each representational system adds to effective communication and helps to convey different meanings of a single mathematical concept.

While verbal and symbolic representa-
tions of the mathematical relations have a long history, more attention has been recently given to "diagrammatic representation" (Diezmann, 1999, pp. 185-190) as a 'visual-spatial' means to represent mathematical ideas, principles, and problem situations. The abilities to recognize, create, interpret, make connections and translate among representations are powerful communication tools for mathematical thinking. Each representational system contributes to effective communication of mathematical ideas by offering certain types of language. The process of representation or representing involves identification, selection and presenting one idea through something else (Seeger, 1998). It might be referred to as a structurally equivalent presentation through pictures, symbols and signs (language and notations). For instance, in order to represent an unknown quantity, one can use the word unknown or a combination of a number symbol and the word, e.g., 2 unknowns; or to draw a line segment or an 'algebra piece' (see fig. 1), or use a letter such as 'm' or 'x'.

It is likely that a fluent and flexible use of multiple representations of "structurally the same" (Dreyfus & Eisenberg, 1996, p. 268) mathematical concept is associated with deep conceptual understanding.

The research in the area of representation has been focused on student generated representation and subsequent impact of these representations on learning mathematical concepts (e.g., Boulton-Lewis & Tait, 1993; Diezmann, 1999; Diezmann & English, 2001; Lowrie, 2001; Outhred & Saradelich, 1997; Swafford & Langrall, 2000; Verschaffel, 1994). Pape and Tochanov (2001) observed that when students generate representations of a concept or while solving problems (as a means of mathematical communication) their natural tendency is to reduce the level of abstraction (given by the problem) to a level that is compatible with their existing cognitive structure. Similarly, Wilensky (1991) suggested that students tend to make the unfamiliar more familiar, and the abstract more concrete. He asserted that students try to "concretize" the concepts they learn to "come to the concept as close as possible" (p. 196). The 'concretizing' can be associated with the construction of an internal representation and may possibly involve the process of reducing the level of abstraction. Therefore, it seems reasonable and warranted to consider
together these two closely related, but distinct ideas: representations and reducing abstraction.

**iii). Abstraction**

The term 'abstraction' has been used to describe both the cognitive process (mental action) of isolating, or 'abstracting' a common feature or relationship observed in a number of things, and the product (mental image or mental object) of such a process. The distinction in meanings is usually ensured by the situated context in which the word 'abstraction' is used.

Czemecka (2006) uses the term abstraction to describe the method for constructing the object of intellectual cognition in general. Abstraction as a mental action separates a property or a characteristic of an object from the object to which it belongs or is linked to and forms a cognitive image or a concept (an abstraction) of the object. Thus, abstraction can be understood as a mental process that promotes the basis of thoughts that allow one to reason. Abstract objects are defined as those that lack certain features possessed by concrete things. The Oxford Desk Dictionary and Thesaurus (1997) suggests that "abstract is what exists in thought or in theory and not in matter or in practice." Turchin (1997) posits that 'abstractness' is a concept in which one does not take into account a specific value or characteristic of the object in consideration, but any of all possible values and characteristics related to the object.

The concept of abstraction in the field of mathematics education research has been examined from different perspectives (e.g., Ferrari, 2003; Frorer et al., 1997; Gray & Tall, 2007; Noss & Hoyles, 1996; Ohlsson & Lehtinen, 1997; Tall, 1991). There is an agreement that mathematics students are continuously involved in the process of abstraction because they are engaged in transformation of their perceptions into mental images by means of different representations. The following notions are essential to examine the processes of transforming prior mental images and developing conceptual understanding: the notion of the degree of abstraction (e.g., Cifarelli, 1998; Heibert & Lefevre, 1986; Mitchelmore & White, 2007; Skemp, 1986; Wilensky, 1991), the notion of adaptation to abstraction (Piaget, 1970; Von Glaserfeld, 1991), and the notion of reducing level of abstraction (Hazzan, 1999; Hazzan & Zaskis, 2005).

Cifarelli's (1988) proposed the levels of reflective abstraction to describe a college students' learning process while they solving algebra word problems. These levels include recognition, representation, structural abstraction, and structural awareness. At the highest level, structural awareness, the student is able to consider problem structures and operate upon them as objects. Skemp (1986), Heibert and Lefevre (1986), and Wilensky (1991) argued that the degree of abstraction is a variable that depends on the student's prior knowledge and subjective way of integrating past experience with new information. If conceptual understanding is defined by the degree of abstraction, then the idea of adaptation to abstraction becomes critical, and the process of building mathematics conceptual understanding
can be viewed as a transition between the levels of abstraction from lower to higher. Hazzan and Zazkis (2005) assert that certain types of concepts are more abstract than others, and that the ability to abstract is an important skill for a meaningful learning of mathematics. Hazzan (1991) posits that the growth in conceptual understanding is manifested by the increased ability to “cope with” a higher degree of abstraction. To describe learners’ behaviors in terms of coping with levels of abstraction, Hazzan (1999) introduced a theoretical framework of reducing level of abstraction. It refers to situations in which students are unable to manipulate the concepts at the level presented in a given problem and therefore, they reduce the level of abstraction of the concepts involved to make these concepts mentally accessible (Hazzan & Zazkis, 2005, p.102).

The transition between the levels of abstractions can be illustrated by the following example: during the process of counting physical objects one abstracts from the properties of these objects and uses linguistic objects or linguistic abstraction, i.e., words to represent the quantity of the physical objects. Next, one uses the number symbol to represent the word that in turn represents the quantity of the physical objects. In algebra, one abstracts from number symbols and uses $x$ to represent all the possible numbers. The following levels of abstraction provide the means to view the process of developing conceptual understanding in algebra. Assume that operating on ‘number words’ which represent certain quantities of real objects is a first level of abstraction (linguistic abstraction). Operating with ‘number symbols’ can be thought as the second level of abstraction, and operating on letters that stand for ‘number symbols’ can be viewed as the third level of abstraction (algebraic abstraction). Thus, one can assert that abstraction in mathematics is an activity of integrating pieces of information (facts) of previously constructed mathematics knowledge and reorganizing them into a new mathematics structure or a new hierarchy. For example, a number line can be viewed as a set of one-unit segments joined together by their ends sequentially. It is also a visual representation of the one-to-one correspondence concept where each point on a number line corresponds to a unique real number and vice versa.

\[
\begin{array}{ccccccc}
-1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

Thus, a single segment \( \bullet \) can be used to represent 1 (fixed number or quantity), as well as ‘1’ can be used to represent a line segment with the length of one unit. Two segments jointed together can represent number 2, etc.

\[
\begin{array}{c}
\bullet \quad \text{One unit (or just 1)} \\
1 \text{ unit} \\
\bullet \bullet \quad \text{Two units (or just 2)} \\
\end{array}
\]

Then the sum of two numbers ‘1’ and ‘2’ can be represented as a line segment which consists of three unit segments

\[
\begin{array}{ccc}
\bullet \quad + \quad \bullet \bullet \quad = \quad \bullet \bullet \bullet \\
1 \text{ unit} & 2 \text{ units} & 3 \text{ units}
\end{array}
\]
Moving to the next level of abstraction, a single segment can represent a fixed quantity which is unknown

\[ x \]

Then two segments of the same length joined together will represent the sum of the two fixed quantities or two unknowns.

\[ 2x \]

Then the sum of one unknown and two unknowns can be described as 3 unknowns.

\[ x + 2x = 3x \]

This example shows the transition from concrete (number system, pictorial aids) to abstract (algebraic symbols). On the other end, when the students are facing with the problem of collecting like terms (e.g., \( x \) and \( 2x \)) where \( x \) represents a variable, they might experience a need to reduce the level of abstraction and to think in numbers and in pictures (e.g., line segments). They can be encouraged to manipulate line segments and numbers (act upon the objects) to find the sum of \( x \) and \( 2x \) as the length of the integral segment and then translate the length of the segment into symbols. The relationship between two processes, developing concept abstraction and reducing the level of abstraction can be illustrated as shows in the figure 2.

Of course, the cognitive processes of abstraction are much more complex than a diagram or a concept map. Any schematization has its natural limitations. As Raymond Nickerson (1986) noted, "Taxonomies are, at best, convenient ways of organizing ideas and should never be taken very seriously. The world seldom is quite as simply divisible into neat compartments as our penchant for partitioning it conceptually would suggest" (p. 358). Nevertheless, it is useful to organize ideas and use pictorial representations for communication purposes. In this connection, it is important to recognize that a line segment representation of a number and/or unknown/variable as any external representation defined by Zhang (1997), provides only certain information, and "stresses some aspects and hide others" (Dreyfus & Eisenberg, 1996, p. 268), thus limited in certain ways. Yet, this representation might be sufficient to use in the process of building the concept of operations with unknown and variable. The need for pictorial representation might become obsolete as the 'object conception' of unknown is formed and developed to the degree of abstraction that (lower level) images are no longer needed to consciously manipulate letters.

Learning algebra, students develop the mental abilities that Inhelder and Piaget (1958) called formal operations. These mental abilities enable the student to deal with highest level of abstraction, i.e., algebraic symbol system. Those students who have not developed formal operations and struggle when dealing with the algebra symbol systems, try to 'reduce' the level of abstraction given by the problem (for example, to solve the equation \( 3x + 4 = 16 \)) to the lower level on which they can
operate, i.e., 'number symbols.' To cope with the problem, these students use a trial and error method replacing a letter (x) with numbers until they find the solution.

Drawn from Piaget's (1970) idea that children first learn about an object by acting upon it and through interaction they eventually understand its nature, theories (Sfard, 1991, 1992; Dubinsky, 1991; Dubinsky & McDonald, 1991) distinguish between a process conception or operational conception and an object conception or structural conception of mathematical principles and notions, and agree that when a mathematical concept is learned, its conception as a process precedes its conception as an object. These theories also suggest that the process conception (e.g., simply following or performing the steps shown by teacher) is less abstract than an object conception (the nature of the concept with its properties, rules and understanding of how and why the rules work). One may conclude that the process conception of a mathematical concept can be interpreted as being on a lower (reduced) level of abstraction than its conception as an object. When students show a tendency to reduce the level of abstraction and work on a lower level of abstraction, it might be hypothesized that although they demonstrate a certain level of process conception, they have not yet developed conceptual understanding.

It seems plausible to assume that every algebra student goes through the process of familiarization with and adaptation to different levels of abstraction. It also seems credible to believe that students are familiarizing and adapting to abstraction at different rate. Wilensky (1991) suggested that the higher the rate of adaptation to abstraction the less the need for reducing the level of abstraction. In this sense, the process of adaptation to abstraction might involve certain behavior manifested in coping with level of abstraction. In other words, when students are unable to manip-
ulate with the level of abstraction (words, numbers, symbols) presented in a given problem, they consciously or unconsciously reduce the level of abstraction of the concepts involved to make these concepts within the reach of their actual mental stage of development.

The above overview of the ideas and assumptions about representations, abstraction and conceptual understanding provided reasonable and sufficient basis for developing the study that offered another perspective on the process of assessing algebra students' conceptual understanding of linear relationship with one unknown.

Method

This multi-year research study was launched to address and explore the interpretative dimensions of educational phenomena (Burns, 2000; Cohen & Manion, 1992; Merrian, 1988) associated with the assessment of levels of middle school students' understanding of linear relationship with one unknown. The internal validity of this research was achieved by data, methodological, and theory triangulation (Burns, 2000; Cohen & Manion, 1992). Data triangulation was ensured by collecting data from different students at different times. Methodological triangulation was achieved through different data collection methods; a comprehensive multifaceted survey and interviews with the students were used to identify issues and themes related to developing conceptual understanding and reducing levels of abstraction. The theory triangulation was related to an epistemological and ontological justification (Merrian, 1988) of the terms representations, adaptation to abstraction level, reducing level of abstraction and conceptual understanding, which are discussed later in the paper from different perspectives. The research inquiries have been addressed through analysis of the survey and analysis of students' thinking process while they were solving problems and explaining their solutions during the interviews.

1) Instruments and sample

The survey, designed by the researcher and described below, consisted of four interrelated parts. It combined a questionnaire and a set of problems, both related to the concept of one-two-step linear equations with one unknown, which are familiar to middle school algebra students. The actual problems were presented in three different modes: words, diagrams, and symbols.

While designing and refining the survey, the researcher conducted three consecutive pilot studies to attain better clarity of the items and directions, to clean up ambiguity in sentences, to check for time of completion, and to address the problems that the students had experienced when taking the pilot survey (McMillan & Schumacker, 1997). The pilot data were collected from three different groups of middle school algebra students in the districts which administered the final version of the instrument during next years. Each pilot provided feedback which enabled the researcher to revise the items. Scores were analyzed for adequate distribution for each item in the instrument (p. 183). The instrument provided consistent
results when repeated (Flower, 1993). Several experts in the field confirmed the content and face validity of the instrument. An inter-item correlation was conducted for each construct to ensure the reliability of the instrument.

1a) Description of the survey and coding system

Part I has 12 items with five optional response scale (always, often, sometimes, rarely, never). The items were aimed to collect information about students’ perceptions, experiences and attitudes towards different representations (words, diagrams, symbols) when they dealt with algebraic problems in general. All items were broken down into several constructs and scoring codes were clustered around students’ preferred mode of representation (for example, the students’ perception of pictures/diagrams, numerical symbols, algebraic symbols, multiple representations).

In Part II four questions each with three choices asked the students to select the response that most closely reflects their current learning practices and most preferable/less preferable mode of thinking (mental habits) when solving linear equations with one unknown. While the items in Part I and Part II might seem redundant they provided a basis to ensure consistency and correspondence of the students’ responses. In other words, if in Part I a student indicated that he/she needs to draw a picture when solving problems, it was expected that in Part II the student would indicate that he/she is comfortable to think in pictures.

Part III illustrated “structurally the same” (Dreyfus & Eisenberg, 1996, p. 268) linear relationship with one unknown posed in three different representations: as a word problem, as a diagram where the unknown number was presented as a line segment, and as an algebraic equation expressed in symbols. Students were not asked to solve the problems and generate solutions, but rather to observe and explain in writing if they recognize the same relationship (the sum of two numbers is 28; one number is 10, what is the other number?) presented in three different modes. The coding system for the items in Part III included 4 codes from 0 to 3. The code 0 was assigned if the students made no attempt and left it blank. The code 1 was assigned to the answers which indicated that the students did not recognize (answered ‘no’) the same relationship presented via three different representations. The code 2 was assigned if the students recognized the relationship presented via three different representations (answered ‘yes’), but did not explicitly verbalize their thinking in a clear and coherent way. The code 3 was assigned if the students recognized the relationship presented via three different representations (answered ‘yes’) and explicitly described the relationship presented via three modes.

Part IV consisted of three sets (A, B, and C) of problems that involved linear relationships with one unknown to be solved using one-two-step addition/subtraction and multiplication/division; each set consisted of three problems and the students were asked to solve each problem. For each problem set in the Part IV a coding system was created. Set A had three problems pre-
sented in words that described three different linear relationships. The coding consisted of 13 codes from 0 to 12. All possible variations were considered and specifically focused on how the problems were solved. For example, it was noted if the students used only numbers and operations (trial and error, no symbols); whether they set up numerical equations or algebraic equations, or whether they used diagrams. If the diagrams were utilized, they were coded depending on the degree of clarity (obvious, apparent or vague).

Set B posed three linear relationships presented in visual form via diagrams. In each diagram numbers and an unknown were illustrated as line segments. The coding system consisted of eleven codes, from 0 to 10, and focused on how the problems presented via diagrams were solved. Each code represented a combination of several variables to document if the students set up numerical or algebraic equation, used correct or incorrect procedure, produced correct or incorrect answer.

Set C contained three linear equations with one unknown represented in symbols. The coding system consisted of seven codes from 0 to 6 to account for whether the students solved the problem correctly/incorrectly using trial and error method; whether they used algebraic method i.e., steps, inverse operations. All survey problems were similar in style and level of difficulty.

The surveys were analyzed to examine the relationship between students' perceptions and use of multiple representations and students' ability to recognize the same linear relationship between unknown and other elements of a problem presented in different modes: words, pictures, symbols. A constant comparison method (Glaser & Strauss, 1967) was applied to develop categories, compare each category, and look for patterns of similar and distinctive attributes.

During the period of four consecutive years four tiers of data were collected from 11 schools in 6 districts, four suburban and two urban with diverse populations of students. The schools were not selected randomly but were approached by the researcher with a request for participation in the study. All the participating schools used the same mathematics curriculum which claims facilitation of reasoning skills and use of multiple representations. The schools administered the survey to all 7 and 8 algebra students \( N_{total} = 753; N_{year1} = 176; N_{year2} = 198; N_{year3} = 207; N_{year4} = 172 \). Each of the four tiers of surveys was analyzed separately and then compared to look for common themes, trends, and tendencies.

The analysis of the survey led the researcher to organize all surveys in three distinct groups to form three major categories, which further induced generation of a hypothesis about the indicators of students' conceptual understanding of linear relationship with one unknown. The categorization of the surveys guided the selection of the students \( N=24 \) for the interviews, eight students from each group. The selection was based on the idea of representative sample from a population, which provides the potential to be able to unitize, categorize and generalize (Glaser & Strauss, 1967; Lincoln & Guba, 1985).
Individual surveys from each category were identified and the interviews with selected students were scheduled in their schools. Prior to the actual interview, the researcher met with the teachers of each student to learn about the student’s ability level, performance, whether they are English Language Learners, etc.

The interviews offered an important layer of evidence about the students’ thinking process and the development of conceptual understanding. First, the students were asked to reflect on their survey responses, and then were presented with similar problems to gain more insight into their thinking process. During the interviews the researcher focused on by-product questions such as: How do students conceive symbolic notations as a mathematical language? To what extent and in what way students use different levels of abstraction to demonstrate different meanings of letters and symbols? Is there a relationship between the form of representation the students use (verbal, diagrammatic, symbolic) and the level of their conceptual understanding; in particular, are there signs of potential progress from a procedural to structural conception of linear relationships and its properties. Students’ verbal explanations were audio-taped and written notes were collected.

Analysis and Formulation of the Three Phase Ranking Framework

As it was indicated above, all the surveys were sorted and organized in three categories, which led to the formulation of the three phase ranking framework for assessing conceptual understanding using multiple representations.

The analysis of data was focused around the definition of algebraic reasoning, how the students used verbal, visual and symbolic representations, ‘coped with’ the level of abstraction presented by the linear equations with one unknown, and whether they recognized structurally the same relationship presented in different modes. The researcher was looking at how the students used different representations to i) extract information from situation, ii) represent the information in other forms, iii) manipulate with representations, and iv) interpret and test the solutions of the linear equations with one unknown.

Phase Zero: The first category of the survey was characterized by several major features. In Part I of the survey 72% of these students reported that they can memorize the rules and 63% can remember the steps. In the Part II of the survey 91% of the students in this category indicated that they think ‘in numbers’, 75% needed to use ‘trial and error’, 19% would prefer to use the rules and only 8% would prefer to use the pictures when solving one-two step linear equations with one unknown. In Part III of the survey this category of students did not recognize that three different modalities (words, diagram and symbols) represent the same relationship (the sum of two numbers is 28, one number is 10, find the other number). In the Part IV, 77% found the unknown number correctly for the problems presented in words, 56% found the unknown length of the segment for the problems presented in a diagram, and 86% found correctly the unknown number in the algebraic equation.
During the interview, each student from this category was encouraged to look at his/her survey response from Part III and verbalize the relationship in each of the three problems presented. The researcher asked questions to stimulate the students’ thinking and recognition that each problem represented structurally the same relationship. The students were also encouraged to create an algebraic equation using symbols (the researcher put a special emphasis on the terms algebraic and symbols) of the relationship stated in words (‘the sum of two numbers is 28, one of the numbers is 10, find the other number’). Six students showed subtraction in a column format

\[
\begin{array}{c}
- & 28 \\
10 & \_ \\
18 & \\
\end{array}
\]

and two wrote the numerical equation, \(28 - 10 = 18\). None of these students produced an algebraic statement that would describe the relationship where a letter stands for unknown number (e.g., \(x + 10 = 28\)). These students apparently had difficulty operating with algebraic sentences (equations) and preferred numerical instantiations (Kieran, 1992, p. 392).

While the students in this category were able to find the correct solutions to the equations presented in symbols and words by using instantiations and/or manipulating with numerical and/or algebraic symbols, quite a few (46%) apparently were having difficulty or were unable to solve the equations that represented the same relationship in a pictorial mode. Further probing revealed a great confusion with diagrams when the students were asked to interpret a given diagram or to create their own. For example, when the students were encouraged to draw a picture that would represent subtraction of 9 from 23, they drew 23 objects (squares or circles) and then drew 9 more of the same objects and said they would take away 9. As a result, their lacking ability and possibly not favorable attitude toward pictorial representation created a barrier for their meaningful learning (Kieran, 1992) and development of conceptual understanding.

Summarizing the above, the persistent need to use numbers (trial and error) when solving linear equations with one unknown presented in a symbolic mode showed that the students reducing the level of algebraic abstraction to numerical abstraction. The dependency on the numbers was an indicator that these algebra students still needed instantiations to operate at the comfort level that numbers provide to them. For these students an algebraic symbol (letter) that stands for an unknown in a linear relationship was an external representation that had not been internalized and integrated into the students’ prior mental structures, and thus it was rejected in favor of the lower level of representation, number symbols, which had already been a tool of their mathematical communication (Javier, 1987; Lesh, Post, & Baker, 1987). A comparison can be drawn between rejecting an external representation and inability to assimilate the external representation into the existing cognitive structure (Piaget, 1970). It is possible that the process conception (computational skills) has been developed in these students to certain degree and they could reproduce and/or
mimic the procedure. However, each time they are faced with a new situation they tried to reduce the level of abstraction to numbers, intuition and trail and error. Their algebraic reasoning had not been advanced to the level of abstraction given by the symbolic representation of the concept therefore it is unlikely that they have achieved conceptual understanding of the linear relationship with one unknown. Let us call this category as Phase 0.

**Phase 1.** The second category of the survey had also several distinguish features. The students in this category reported that they didn’t need to try the numbers when solving equations. This can be interpreted that they didn’t need to reduce the level of abstraction given by the problem. They indicated that could ‘think in symbols’ (86%) and rather preferred symbols to diagrams and word problems. The students also reported that they preferred using the steps (82%), and were able to memorize rules (93%). Many of them indicated that didn’t need or want to use pictures/diagrams (68%) however when presented with a diagram that displayed a linear relationship with one unknown were able to find the length of the unknown segment. All showed proficiency in solving the same type of linear equations with one unknown provided by the survey and during the interview. Nevertheless, this category of students either didn’t recognize (answered ‘no’ or left the blank) that the three modes (words, diagram and symbols; survey Part III) represented structurally the same relationship. Some students made an attempt to describe their thinking but could not produce clear written explanation and explicitly describe the connections between the representations. During the interviews, when they were given a picture (see fig. 3), the students found the length of the unknown segment, however when asked to write an algebraic statement that would describe the picture, they produced the same type of numerical statement (e.g., $23 - 9 = 14$) as the first category of students.

These students knew and could verbalize the steps when solving equations, but the explanations were lacking true understanding, i.e., flexibility of reasoning. Their behavior could have been described as a well rehearsed acting upon fixed rules (i.e., isolate the unknown, use inverse operation2). Further probing revealed that these students’ actions were more mechanical than rooted in logic, which supports the idea that middle school algebra students, particularly those who are in transition.

In summary, the students in this category were able to extract the correct relationship between unknown and known presented via a word problem, manipulate with symbols, correctly verbalize the steps and show certain degree of proficiency without reducing the level of abstraction given by the problems. For them the steps were most certainly the object abstraction they could manipulate without reducing. It can be said that these students were at the Cifarelli's (1988) stage of structural abstraction, and perhaps were able to blend the process conception and the object conception (Tall, 2008). However, these factors were insufficient to draw the conclusion that the students achieved conceptual understanding. While fluent process skills are important, and these skills can be viewed as prerequisite and a foundation for development of flexible thinking, they cannot be equated with flexible thinking. The most critical issue was that the students in this category were not able to make connections between different representations (words, pictures, symbols) that characterize structurally the same linear relationship. Let us call this category Phase 1 or the phase of adaptation to algebraic operations.

It is important to stress that the researcher does not claim that the Phase 0 and Phase 1 are sequential and represent a hierarchy. It is possible that the Phase 1 students never experienced the need in using trial and error and instantiations, other words had not been categorized as Phase 0 students.

Phase 2. The major characteristic of the students in the third category was that they recognized (answered "yes"; survey Part III), and explained in relatively clear written language that three different modes represented structurally the same linear relationship. They solved all survey problems correctly using algebraic methods. When interviewed these students showed flexible use of each mode of representation. They made connections between the representations, logically interpreted and translated among representations, and use algebraic symbols as objects. When probed with questions, they showed the ability to explain the full meaning of the concept of unknown, its relationship to the operations, and demonstrated the ability to discern, infer and interpret different representations of the linear relationship. It was evident that these students were able to manipulate with different representations and demonstrated flexible thinking of the properties of the linear equations with one unknown. When asked to describe algebraically the diagrams from the survey Part IV, problem set B, they produced correct equations (e.g., \(2x + 8 + 5 = 57\)). When asked to draw a diagram that would represent given equations from the survey Part IV, problem set C (i.e., \(m + 13 = 38\), \(2n = 26\), \(3x + 2 + 4 = 27\)), they generated valid and apparent pictorial representations. According to Cifarelli (1988) these students operated at the higher level of reflective abstraction, structural awareness. They had blended the process conception and the object conception (Tall, 2008) by showing the ability
to think and act upon the problems’ structures as objects.

Given the above, it is plausible to assert that the most significant indicator of conceptual understanding of linear relationship with one unknown is the ability to recognize structurally the same relationship presented in different modalities and provide an explicit explanation. The students, who recognized the structure of the relationship from each and all representations (verbal, diagrammatic, numerical and symbolic), could interpret, connect and translate among the representations of this relationship with confidence. When the students revealed understanding of meaning of solutions and the properties of the linear relationship, knew how to extract information from situation presented in different forms and how to manipulate representations. It is likely that they conceptualized basic computational strategies, meanings of unknown, variable and expressions with unknown. Let us call this category Phase 2 or conceptual understanding phase.

Discussion and Implications for Teaching

1. Using the three phase ranking framework to assess conceptual understanding

The three phase ranking framework related to the linear relationship with one unknown emerged from this research is based on the idea of levels of abstractions or hierarchical abstraction (Ciffareli, 1988; Hiebert & Lefevre, 1986; Skepm, 1986). Acknowledging the distinction between the levels seems beneficial and practical since it suggests a functional mental tool for mathematics educators to organize thinking around the processes of developing conceptual understanding.

The framework is not overwhelmed with a massive amount of phases (while of course the cognitive processes around the development of conceptual understanding are much more perplexing and intricate than just a schema or taxonomy), thus accessible and manageable. It provides a recommendation for teachers to make a relatively reliable judgment as to whether each student has developed conceptual understanding of linear equations with one unknown and can be easily interpreted into classroom environment. The very fact that a student recognizes that the concept/relationship can be presented in different modes might serve as an indicator that the student is advancing to the next level of learning, namely from procedural skills to structural skills. The framework helps to ascertain whether students are building conceptual understanding instead of efficiently repeating the process. It provides the teachers with insight into the levels of each student’s thinking process and the student’s way of operating with abstractions inherent in algebra. Such information is essential to planning instruction for naturally diverse population of students with wide range of abilities, learning preferences and attitudes. The researcher has already collected anecdotal data on the framework applicability to the linear relationship with two variables which is rich with representations (e.g., words problems, tables, graphs, formulas). The framework has also been used by teachers to assess upper elementary grade students’ under-
standing of the laws of operations via word problems, pictures with line segments and letter symbol representation.

2. By-product observations: Attitudes toward pictorial, verbal and symbolic representations

The analysis of this research data shows that for the Phase 0 and Phase 1 students, the relationship between known and unknown presented in symbols and words were more obvious than the same relationship presented in a picture. A straightforward direction (i.e., solve for \( x \), if \( x + 10 = 28 \)), practically did not require translation as every interviewed student noted. All the students readily and without hesitation either used instantiations or steps to find the solution. Also, it appeared that for these students grasping pictorial representation (i.e., find the length of the segment ‘\( x \)’) was more problematic than understanding verbal representation (i.e., the sum of two numbers is 28, one number is 10, find the other number). The interpretation of both the word problem and a picture involved various cognitive operations and more intellectual power than simply performing steps that students learned, memorized and practiced. It appeared that pictures, created by someone else, could be more complex visual external representations which involve some coded ideas that students have to decipher to be able to adapt to their unique preexisting cognitive structures (Piaget, 1970; von Glasersfeld, 1991). These students were required to recognize and distinguish from the picture the actual valid and correct relationship that were not explicit and entailed more analysis than the same relationship formulated in words. Verbal description (representation) of the picture called for coherent and clear thinking to be expressed in a precise language to make sense. It might be useful to bring the analogy to concept mapping which is a unique to the person’s vision of the ideas and their connections (Novak & Musonda, 1991; Novak & Wandersee, 1990). Reading a concept map requires the same mental effort as reading a pictorial representation of an algebraic problem.

3. Final ruminations

It might be hypothesized that the students who do not demonstrate conceptual understanding have not been exposed to the ‘culture’ of multiple representations. It seems unlikely that students can develop conceptual understanding in mathematics without being systematically encouraged to observe all aspects, meanings and subtleties of the concepts, principles and ideas and without being stimulated to recognize and use multiple representations of the same concept. It would significantly impoverish students’ mathematics learning if they see only limited aspects rendered by one representation. Objectively, each representational mode has its limitations and restrictions as well as notable advantages. A number symbol system is apparently an effective and compact way to represent quantities and is a prerequisite for algebraic symbolism. While, numbers provide instantiations to build up the more general case, they might actually impede comprehending the principals and properties of algebraic concepts.
as has been shown in this study. Verbal representations usually associated (but not limited) with word problems, require students to have the linguistic skills that would help them to interpret sometimes ambiguous verbal statements, yet, verbal representations provide a natural context and natural everyday mode of communication. Pictorial, diagrammatic or visual representations can be as confusing as they are powerful. The algebraic symbol system provides an avenue for expressing mathematical principles, concepts and ideas in general forms by using mathematical models. It is the only system that offers opportunity to logically investigate, justify, generalize and prove mathematical hypotheses. Yet, the algebraic symbol system is too sophisticated to some students. All the representations integrated into teaching and learning would contribute to the development of a big mental picture of the concepts studied. Even the student who is advanced and capable, has developed strong procedural skills, and achieved some level of operational conception will benefit from learning multiple tools to communicate his/her cognitive skills.

The idea of multiple representations hardly needs advocacy to support its power, significance and vitality for mathematics and mathematics teaching. Mathematics is a science of patterns and multiple representations of these patterns. Undeniably, multiple representations should be an integral part of teaching and learning mathematics. Teaching through the multiple representations would require teachers to have a solid foundation in mathematical content, strong skills in a structural analysis of the concepts and tasks as well as sound knowledge in effective planning. Teachers skilled in seeing the big picture of the concept and small links between the sub-concepts are likely to be able to integrate multiple representations into their instructional practices and to use representations when teaching all students, rather than for remediation purposes only. It cannot be assumed that if the ‘culture’ of multiple representations has not been created in the classroom by the teacher, the student will intuitively and suddenly come up with them. As discussed in this paper, multiple external representations presented to the students are likely to stimulate in students the development of conceptual understanding of mathematics. With the encouragement and support of teachers students will be able to internalize and integrate multiple representations into their cognitive structure and use the representations as communication tool. For this to be happen the idea of a ‘culture of multiple representations’ should be integrated and built into teacher preparation programs and teacher professional development. More research is needed to better understand the connections between a teacher’s content knowledge, teacher’s attitude toward multiple representations, and the teacher’s use of multiple representations in the classroom.
References


Cifarelli, V. V. (1988). The role of abstraction as a learning process in mathematical problem solving. Doctoral dissertation, Purdue University, Indiana.


